MINIMAL COMMUTATOR PRESENTATIONS IN FREE GROUPS

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Abstract. We present a fast algorithm for computing the commutator length and an algorithm that gives a minimal commutator presentation of an element $w \in [F,F]$, where F is a free group. Moreover, we give a algorithm that defines whether the element a commutator or not.

Introduction

Let G be a group and $[G,G]$ be its commutator subgroup which is generated by commutators $[g,h] = g^{-1}h^{-1}gh$. Every element from the commutator subgroup can be presented as a product of commutators. Such presentation of an element $g \in [G,G]$ that contains the least possible number of commutators is called a minimal commutator presentation of g. The number of commutators in a minimal presentation of g is called the commutator length of g and is denoted by $\mathsf{cl}(g)$. For example if $G = F(x,y)$ is the free group on two generators, then $\mathsf{cl}([x,y][x^{-1},y^{-1}]) = 1$ and $\mathsf{cl}([x,y]^3) \le 2$ because

$$[x,y][x^{-1},y^{-1}] = [yx^2, {}^{yx}y], \qquad [x,y]^3 = [x^{-1}yx, x^{-2}yxy^{-1}][yxy^{-1}, y^2].$$

In this paper we work only with the free group $F = F(x_1, \dots, x_N)$.

The first algorithm for computing commutator length was constructed by Goldstein and Turner in [4]. Culler in [3] presented another algorithm, which applies not only to free groups but also to free products. It was also stated that in $F(x,y)$ for any $m \ge 1$,

$$\mathsf{cl}([x,y]^n) = \lfloor n/2 \rfloor + 1.$$

Another algorithm for computing commutator length comes from [6]. The first purely algebraic algorithm was offered by V.G.Bardakov in [1]. To our knowledge, there was no purely algebraic (i.e. programmable) algorithm for computing a minimal commutator presentation.

The goal of this paper is to present a fast algorithm for computing the commutator length and an algorithm that gives a minimal commutator presentation of an element $w \in [F,F]$, where F is a free group. Moreover, we give a algorithm that defines whether an element is a commutator or not. This algorithm is much faster than the algorithm which is based on Wicks' theorem [5]. These algorithms are based on Theorem 5.1 which was proved using Bardakov's work [1].

The basic remark for our work is the following. If

$$w = w_1 a^{-1} w_2 b^{-1} w_3 a w_4 b w_5,$$

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then

$$w = [w_1 w_4 w_3 a w_1^{-1}, w_1 w_4 b w_2^{-1} w_3^{-1} w_4^{-1} w_1^{-1}] w_1 w_4 w_3 w_2 w_5.$$

Moreover, the Corollary 5.2 states that w is a commutator if and only if there exists a presentation without cancellation $w = w_1 a^{-1} w_2 b^{-1} w_3 a w_4 b w_5$, where $a, b \in \{x_1^{\pm 1}, \dots, x_N^{\pm 1}\}$ such that $w_1 w_4 w_3 w_2 w_5 = 1$. And in this case $w = {}^{w_1 w_4}[w_3 a w_4, b w_2^{-1} w_3^{-1}]$. This corollary gives an obvious algorithm that defines whether an element is a commutator or not. This is a corollary of Theorem 5.1 that states the following. For any $w \in [F, F]$ there exists a presentation without cancellations $w = w_1 a^{-1} w_2 b^{-1} w_3 a w_4 b w_5$ such that $a, b \in \{x_1^{\pm 1}, \dots, x_N^{\pm 1}\}$ and

$$\mathsf{cl}(w_1 w_4 w_3 w_2 w_5) = \mathsf{cl}(w) - 1.$$

We do not have a good theoretical estimation of the complexity of our algorithm but in practice we see that our algorithm is much faster than the algorithm which is based on Bardakov's theorem. We wrote computer programs using these two algorithms and we see that the program which is based on our algorithm computes the commutator length for a random word $w \in [F_2, F_2]$ of length 24 in a moment but the program that is based on Bardakov's work computes the commutator length in several minutes. All the computations were done on our home computers.

Using our algorithm we get several minimal commutator presentations of $[x, y]^3$:
$[x,y]^3 = [yx^{-1}y^{-1}x^2, yx^{-1}y^{-1}xyx^{-1}yxy^{-1}][yxy^{-1}, y^2]$;
$[x,y]^3 = [x, yxy^{-1}x^{-1}y][y^{-1}xy^{-1}x^{-1}y, y^{-1}x^2y]$;
$[x,y]^3 = [yx^{-1}y^{-1}x^2, yx^{-1}y^{-1}xyxy^{-1}x^{-1}yx^{-1}yxy^{-1}][x^2y^{-1}, yx^{-1}yx^{-1}]$;
$[x,y]^3 = [x^{-1}y^{-1}xy^2x^{-1}y^{-1}x^2y^{-1}x^{-1}yx, x^{-1}y^{-1}xy^2x^{-1}y^{-1}xyx^{-1}yxy^{-2}x^{-1}yx][x^{-1}y^2x, x^{-1}yx^{-1}y^{-1}x]$;
$[x,y]^3 = [x^{-1}y^{-1}xyxy^{-1}x^{-1}yx, x^{-1}y^{-1}xy^2xy^{-1}x^{-2}yx][x^2, xyx^{-1}]$;
$[x,y]^3 = [x^{-2}y^{-1}xy^2x, x^{-2}y^{-1}xyx^{-1}y^{-1}x^{-1}yx^2][x^2, y]$;
$[x,y]^3 = [x^{-1}yx, x^{-2}yxy^{-1}][yxy^{-1}, y^2]$;
$[x,y]^3 = [x^{-1}y^{-2}xyx^{-2}yx, x^{-1}y^{-2}xyx^{-1}y^{-1}xy^{-1}x^{-1}y^2x][x, y^2]$;
$[x,y]^3 = [x^{-1}y^{-1}x^{-1}yx, x^{-1}y^{-2}x^{-1}yx^2][x^2, y]$;
$[x,y]^3 = [x^{-1}y^{-1}x^2yx^{-1}y^{-2}x^{-1}yx, x^{-1}y^{-1}x^2yx^{-1}y^{-1}xyxy^{-1}x^{-2}yx][x^2, xyx^{-1}]$;
$[x,y]^3 = [x^{-1}y^{-1}xy^{-1}x^{-1}yx, x^{-1}y^{-1}x^2y^{-1}x^{-1}y^2x][x, y^2]$.

The paper is organised as follows. In Section 1 we give some notations and conventions. In Section 2 we describe the algorithms without any proofs. In Section 3 we recall Bardakov's results in appropriate notation. In Section 4 we prove some preliminary results about permutations. The main of them is Proposition 4.4. In Section 5 we prove the main statement (Theorem 5.1), using Proposition 4.4 and Bardakov's theorem (Theorem 3.1).

1 Notations and conventions

Let $\{x_1, \dots, x_N\}$ be a set with N elements. We call it **alphabet** and its elements we call letters. A **word** is a sequence $W = a_1 a_2 \dots a_n$, where $a_i \in \{x_1^{\pm 1}, \dots, x_N^{\pm 1}\}$. We distinguish words and elements of the free group $F := F(x_1, \dots x_N)$, which are classes of words. Words are denoted by capital letters $W, U, \dots$ but elements of the free group are denoted by $w, u \dots$. Let $w_1, \dots w_n, w \in F \setminus \{1\}$ and $W_1, \dots, W_n, W$ be the corresponding reduced words. The element w is said to be a **product without cancellation** of the elements $w_1, \dots, w_n$ if $W = W_1 \dots W_n$. For any word W there exists a unique reduced word $\mathsf{red}(W)$ that represents the same element in F. A reduced word $W = a_1 \dots a_n$ is said to by **cyclically reduced** if $a_1 \neq a_n^{-1}$. For any word W we can obtain a cyclically reduced word by applying several times the operation of deleting

the letters a_1 and a_n if $a_1 = a_n^{-1}$ to the word $\mathsf{red}(W)$. We denote this cyclically reduced word by $\mathsf{cycl.red}(W)$. It represents the same conjugacy class in F.

The commutator of $w, u \in F$ is given by $[w, u] = w^{-1}u^{-1}wu$. The commutator subgroup of F is denoted by $[F, F]$. The presentation of $w \in [F, F]$ as $w = [w_1, w_1'][w_2, w_2'] \cdot \ldots \cdot [w_l, w_l']$ is said to be **commutator presentation** of w. The minimal length of commutator presentations is called the **commutator length** of w and denoted by $\mathsf{cl}(w)$.

If X is a finite set, we denote by $S(X)$ the group of permutations of X. If $X = \{1, \ldots, n\}$, then $S(X)$ is denoted by S_n. If $\sigma \in S(X)$ and $x \in X$, the σ-**orbit** of $x \in X$ is the set $\{\sigma^n(x) \mid n \in \mathbb{Z}\}$. The number of σ-orbits is denoted by $o(\sigma)$. A permutation $\pi \in S(X)$ is said to be involution if $\pi^2 = 1$. For an involution $\pi \in S_n$ we set

$$v(\pi) := o((1, \ldots, n)\pi),$$

where $(1, \ldots, n) \in S_n$ is the long cycle.

2 Algorithms

In this section we describe algorithms for computing the commutator length and a minimal commutator presentation of an element $w \in [F, F]$ that is based on Theorem 5.1.

First we describe the algorithm for computing the commutator length of an element $w \in [F, F] \smallsetminus \{1\}$. Let $W = a_1 \ldots a_n$ be the corresponding reduced word where $a_i \in \{x_1^{\pm 1}, \ldots, x_N^{\pm 1}\}$. Denote by $C_1(W)$ the set of words of the form $\mathsf{cycl.red}(W_1W_4W_3W_2W_5)$ such that there is a presentation as a product without cancellations $W = W_1a^{-1}W_2b^{-1}W_3aW_4bW_5$. Further we define a sequence of sets of words $C_1(W), \ldots, C_l(W)$ by recursion as follows. Assume that $1 \notin C_i(W)$. If there are words that are cyclically equivalent to each other in $C_i(W)$ we leave only one from a class of cyclical equivalence and get a new smaller set $C_i'(W) \subseteq C_i(W)$ that does not have cyclically equivalent words. And define $C_{i+1}(W) = \bigcup_{U \in C_i(W)} C_1(U)$. The procedure stops when $1 \in C_l(W)$. It means that $\mathsf{cl}(W) = l$.

The algorithm for computing a minimal commutator presentation is similar. Denote by $D_1(W)$ the set of 3-tuples of the form $\big(\mathsf{red}(W_1W_4W_3aW_1^{-1}),\ \mathsf{red}(W_1W_4bW_2^{-1}W_3^{-1}W_4^{-1}W_1^{-1}),\ \mathsf{red}(W_1W_4W_3W_2W_5)\big)$ such that there is a presentation as a product without cancellations $w = w_1a^{-1}w_2b^{-1}w_3aw_4bw_5$. Further we define a sequence of sets $D_1(W), \ldots, D_l(W)$ by recursion, where $D_i(W)$ consists of $2i{+}1$-tuples of words. Assume that $(\ldots, 1) \notin D_i(W)$. And define $D_{i+1}(W) = \bigcup_{(U_1, \ldots, U_{2i+1}) \in D_i(W)} \{(U_1, \ldots, U_{2i}, U', U'', U''') \mid (U', U'', U''') \in D_1(U_{2i+1})\}$. The procedure stops when $(U_1, \ldots, U_{2l}, 1) \in D_l(W)$. It means that $w = [U_1, U_2][U_3, U_4] \cdot \ldots [U_{2l-1}, U_{2l}]$. Here $l = \mathsf{cl}(w)$.

3 Bardakov's theorem

This section is devoted to describing Bardakov's theorem [1] in slightly different notation.

Let

$$W = a_1 \ldots a_n$$

be a (probably non-reduced) word that presents an element from a commutator subgroup, where $a_i \in \{x_1^{\pm 1}, \ldots, x_N^{\pm 1}\}$. A **pairing** on the word W is an involution $\pi \in S_n$ such that

$$a_{\pi(i)} = a_i^{-1}.$$

The set of pairings on W is denoted by $\mathsf{P}(W)$. Since W represents an element from the commutator subgroup, the number of occurrences of a letter a is equal to the number of occurrences of a^{-1}, and hence $\mathsf{P}(W) \neq \varnothing$. A pairing is said to be **extreme**, if $v(\pi)$ is maximal possible number for $\pi \in \mathsf{P}(W)$.

Theorem 3.1 ([1])**.** *Let a word $W = a_1 \dots a_n$ represent an element $w \in [F, F]$ and π be an extremal pairing on W. Then*

$$\mathsf{cl}(w) = \frac{1 - v(\pi)}{2} + \frac{n}{4}.$$

Remark 3.2. Bardakov in [1] considers only the case of a cyclically reduced word W but he does not use it in the proof.

4 Preliminary results about permutations

4.1 A permutation induced on a subset

Let X be a finite set and $Y \subseteq X$. For $\sigma \in S(X)$ and $y \in Y$ we denote by $m(y) \geq 1$ the least positive integer such that $\sigma^{m(y)}(y) \in Y$. Then

$$\sigma_Y(y) = \sigma^{m(y)}(y).$$

It is easy to see that $\sigma_Y \in S(Y)$. Consider the set $Y_\sigma \supset Y$ given by

$$Y_\sigma = \{x \in X \mid \exists i : \sigma^i(x) \in Y\}.$$

We denote by

$$f_\sigma : Y_\sigma \longrightarrow Y$$

the map given by $f_\sigma(x) = \sigma^{m_0(x)}(x)$, where $m_0(x) \geq 0$ is the minimal non-negative integer such that $\sigma^{m_0(x)}(x) \in Y$. Note that $f_\sigma(y) = y$ for all $y \in Y$, and the function f_σ depends only on the restriction of σ on $\bar{Y} \setminus Y$. If we denote by $\sigma|_Y : Y \to Y_\sigma$ the restriction of σ, we get

$$\sigma_Y = f_\sigma \, \sigma|_Y.$$

Now we generalise the definition of σ_Y to the case of injective map $\alpha : Y \to X$. Let X, Y be finite sets and $\alpha : Y \to X$ be an injective map. For $\sigma \in S(X)$ and $y \in Y$ we denote by $m(y) \geq 1$ the least positive integer such that $\sigma^{m(y)}(\alpha(y)) \in \alpha(Y)$. Then

$$\sigma_\alpha(y) = \alpha^{-1}(\sigma^{m(y)}(\alpha(y))).$$

It is easy to see that $\sigma_\alpha \in S(Y)$,

$$\alpha\sigma_\alpha = \sigma_{\alpha(Y)}\alpha.$$

and if $\alpha : Y \to X$ is an identical embedding, then $\sigma_\alpha = \sigma_Y$.

Proposition 4.1. *Let $\alpha : Y \to X$ be an injective map, $\sigma, \tau \in S(X)$ such that $\tau(\alpha(Y)) = \alpha(Y)$ and $\bar{\tau} \in S(X)$ is given by $\bar{\tau}(x) = \tau(x)$ if $x \in X \setminus Y$, and $\bar{\tau}(x) = x$ if $x \in Y$. Then $\tau_\alpha(y) = \alpha^{-1}\tau(\alpha(y))$ for all $y \in Y$ and*

$$(\sigma\tau)_\alpha = (\sigma\bar{\tau})_\alpha \, \tau_\alpha.$$

Proof. Without loss of generality we assume that α is an identical embedding. It is obvious that $\tau_Y = \tau|_Y$. Since $(\sigma\tau)_Y = f_{\sigma\tau}(\sigma\tau)|_Y = f_{\sigma\tau}\,\sigma|_Y\,\tau_Y$ and $(\sigma\bar{\tau})_Y\tau_Y = f_{\sigma\bar{\tau}}(\sigma\bar{\tau})|_Y\,\tau_Y = f_{\sigma\bar{\tau}}\,\sigma|_Y\,\tau_Y$, it is sufficient to prove that $f_{\sigma\tau} = f_{\sigma\bar{\tau}}$. It follows from the fact that the restrictions of the functions $\sigma\tau$ and $\sigma\bar{\tau}$ on $X \setminus Y$ coincide. □

Lemma 4.2. *Let $\alpha : Y \to X$ be an injective map, $\sigma \in S(X)$ and A is an σ-orbit. Then either $A \cap \alpha(Y) = \varnothing$ or $\alpha^{-1}(A)$ is a σ_Y-orbit.*

Proof. Without loss of generality we assume that α is an identical embedding. Assume $A \cap Y \neq \varnothing$. If $y \in A \cap Y$, then $A = \{\sigma^i(y) \mid i \geq 1\}$. Consider the increasing sequence of all numbers $1 \leq m_1 < m_2 < m_3 < \dots$ such that $\sigma^{m_i}(y) \in Y$. Then $A \cap Y = \{\sigma^{m_i}(y) \mid i \geq 1\}$ and $\sigma^{m_i}(y) = \sigma_Y^i(y)$. It follows that $A \cap Y = \{\sigma_Y^i(y) \mid i \geq 1\}$ is an orbit of y with respect to the permutation σ_Y. □

Corollary 4.3. *Let $\sigma \in S(X)$ such that $A \cap \alpha(Y) \neq \varnothing$ for any σ-orbit A. Then the map given by $A \mapsto \alpha^{-1}(A)$ is a bijection between the set of σ-orbits and the set of σ_α-orbits. In particular, $o(\sigma) = o(\sigma_\alpha)$.*

4.2 Orbits and induced involutions

Consider $n \geq 5$ and $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$. Denote by

$$\alpha : \{1, \dots, n-4\} \longrightarrow \{1, \dots, n\}$$

the unique strictly monotonic map whose image does not contain i_1, i_2, i_3, i_4. Consider $\gamma \in S_{n-4}$ given by

$$\gamma(x) = \begin{cases} x, & \text{if } x \leq i_1 - 1, \\ x + i_4 - i_2 - 2, & \text{if } i_1 - 1 < x \leq i_2 - 2, \\ x + i_1 + i_4 - i_3 - 1, & \text{if } i_2 - 2 < x \leq i_3 - 3, \\ x + i_1 - i_3 + 2, & \text{if } i_3 - 3 < x \leq i_4 - 4, \\ x, & \text{if } i_4 - 4 < x. \end{cases} \tag{4.1}$$

Proposition 4.4. *Let $\pi \in S_n$ be an involution such that $\pi(i_1) = i_3$ and $\pi(i_2) = i_4$. Then in the above notation $v(\pi) = v(\gamma\pi_\alpha\gamma^{-1})$.*

Proof. For simplicity we set $\sigma := (1, \dots, n)\pi$ and

$$\tilde{C} = (1, \dots, n)(i_1, i_3)(i_2, i_4),$$

where (i_1, i_3) and (i_2, i_4) are transpositions. It is easy to check that $\tilde{C}$ is a long cycle. Proposition 4.1 implies that

$$\sigma_\alpha = \tilde{C}_\alpha \pi_\alpha. \tag{4.2}$$

Since $\tilde{C}$ is a long cycle, using Lemma 4.3, we get that $\tilde{C}_\alpha$ is a long cycle too. Simple computations show that

$$\tilde{C}_\alpha = (1, \dots, i_1 - 1, i_3 - 2, \dots, i_4 - 4, i_2 - 1, \dots, i_3 - 3, i_1, \dots, i_2 - 2, i_4 - 3, \dots, n - 4).$$

This description works even if $i_{k+1} = i_k + 1$ for $k = 1, 2, 3$. In this case the line $i_k - k + 1, \dots, i_{k+1} - k - 1$ in the description of the cycle $\tilde{C}_\alpha$ is empty. If $i_4 = n$, then the line $i_4 - 3, \dots, n - 4$ is empty, and if $i_1 = 1$, then the line $1, \dots, i_1 - 1$ is empty. Since $\tilde{C}_\alpha$ is a long cycle, it is conjugated to the long cycle $(1, \dots, n-4)$. One can check that

$$\tilde{C}_\alpha = \gamma^{-1}(1, \dots, n-4)\gamma.$$

Using the equation (4.2) we obtain

$$\sigma_\alpha = \gamma^{-1}(1,\dots,n-4)\gamma\pi_\alpha. \tag{4.3}$$

It is sufficient to prove that $o(\sigma) = o((1,\dots,n-4)\gamma^{-1}\pi_\alpha\gamma)$. Since $o((1,\dots,n-4)\gamma\pi_\alpha\gamma^{-1}) = o(\gamma^{-1}(1,\dots,n-4)\gamma\pi_\alpha) = o(\sigma_\alpha)$, we need to prove that $o(\sigma) = o(\sigma_\alpha)$. Then by Corollary 4.3 it is enough to check that all σ-orbits intersect $\{1,\dots,n\} \smallsetminus \{i_1,i_2,i_3,i_4\}$. In other words, we need to check that the σ-orbit of i_k do not lie in the set $\{i_1,i_2,i_3,i_4\}$ for $k = 1,2,3,4$. Check it. Further in the proof we take sums modulo n. Then $\sigma(i_k) = i_{k+2}+1$. If $i_{k+2}+1 \neq i_{k+3}$, we done. Assume that $i_{k+2}+1 = i_{k+3}$. Then $\sigma^2(i_k) = \sigma(i_{k+3}) = i_{k+1}+1$. If $i_{k+1}+1 \neq i_{k+2}$, we done. Assume that $i_{k+1}+1 = i_{k+2}$. Then $\sigma^3(i_k) = \sigma(i_{k+2}) = i_k+1$. If $i_k+1 \neq i_{k+1}$, we done. Assume that $i_k+1 = i_{k+1}$. Then $\sigma^4(i_k) = \sigma(i_{k+1}) = i_{k+3}+1$. If $i_{k+3}+1 \neq i_k$, we done. Assume that $i_{k+3}+1 = i_k$. Then $i_k = i_{k+3}+1 = i_{k+2}+2 = i_{k+1}+3 = i_k+4$. It follows that 4 divisible by n but it contradicts to the assumption $n \geq 5$. □

5 The main theorem

Note that if $w = w_1a^{-1}w_2b^{-1}w_3aw_4bw_5$, then

$$w = [w_1w_4w_3aw_1^{-1}, w_1w_4bw_2^{-1}w_3^{-1}w_4^{-1}w_1^{-1}]w_1w_4w_3w_2w_5. \tag{5.1}$$

This equation is the key equation for computing the minimal commutator presentation of an element of $[F,F]$.

Theorem 5.1. *Let $F = F(x_1,\dots,x_N)$ and $w \in [F,F]$. Then there exists a presentation of w without cancellations*

$$w = w_1a^{-1}w_2b^{-1}w_3aw_4bw_5$$

such that $a,b \in \{x_1^{\pm1},\dots,x_N^{\pm1}\}$ and

$$\mathsf{cl}(w_1w_4w_3w_2w_5) = \mathsf{cl}(w) - 1.$$

Proof. Let w be presented by a reduced word $W = a_1\dots a_n$, where $a_i \in \{x_1^{\pm1},\dots,x_N^{\pm1}\}$, and $\pi \in S_n$ be an extreme pairing on W. Consider the maximal number $i_2 \in \{1,\dots,n\}$ such that $\pi(i_2) > i_2$. Set $i_4 := \pi(i_2)$. Then $b := a_{i_4} = a_{i_2}^{-1}$. Since the word W is reduced, there exists i_3 such that $i_2 < i_3 < i_4$, $a_{i_3} \neq a_{i_2}$ and $a_{i_3} \neq a_{i_2}^{-1}$. Set $i_1 = \pi(i_3)$ and $a = a_{i_3} = a_{i_1}^{-1}$. It is easy to see that $i_1 < i_2 < i_3 < i_4$. It follows that the word W is presented as the following product without cancellations

$$W = W_1a^{-1}W_2b^{-1}W_3aW_4bW_5,$$

where $W_1 = a_1\dots a_{i_1-1}$, $W_2 = a_{i_1+1}\dots a_{i_2-1}$, $W_3 = a_{i_2+1}\dots a_{i_3-1}$ $W_4 = a_{i_3+1}\dots a_{i_4-1}$, $W_5 = a_{i_4+1}\dots a_n$. Further, we put $\tilde W = W_1W_4W_3W_2W_5$ and let $\tilde w$ be the corresponding element in the free group. It is sufficient to prove that $\tilde l = l-1$, where $l = \mathsf{cl}(w)$ and $\tilde l = \mathsf{cl}(\tilde w)$. Using (5.1) we obtain $\tilde l \geq l-1$. Therefore it is enough to show that $\tilde l \leq l-1$. Let $\alpha : \{1,\dots,n-4\} \to \{1,\dots,n\}$ be the unique injective monotonic map whose image does not contain $\{i_1,i_2,i_3,i_4\}$, and let $\tilde\pi = \gamma\pi_\alpha\gamma^{-1} \in S_{n-4}$ be the permutation from Proposition 4.4. Using that π is a pairing on W, it is easy to check that $\tilde\pi$ is a pairing on $\tilde W$. Proposition 4.4 implies $v(\tilde\pi) = v(\pi)$. Therefore,

$$\tilde l \leq \frac{1-v(\tilde\pi)}{2} + \frac{n-4}{4} = \frac{1-v(\pi)}{2} + \frac{n}{4} - 1 = l-1.$$

□

Corollary 5.2. *Let $w \in [F, F]$. Then w is a commutator if and only if w can be presented as a product without cancellation*

$$w = w_1 a^{-1} w_2 b^{-1} w_3 a w_4 b w_5$$

so that

$$w_1 w_4 w_3 w_2 w_5 = 1.$$

In this case $w = {}^{w_1 w_4}[w_3 a w_4, b w_2^{-1} w_3^{-1}]$.

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